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String free energy, Hagedorn and gauge/string duality

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ABSTRACT: We give examples of modeling the string free energy whose behavior mimics that of QCD: a power-law at high temperature and an exponential decrease at low temperature. Although the effective description is in terms of strings, no limiting temperature exists, as expected for a crossover.

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Contents

1 The problem

Since the late sixties and early seventies, a limiting temperature also known as the Hagedorn temperature $T_{\rm H}$ [1] has attracted great interest in string theory [2]. At zero temperature, strings show an exponential growth in the density of states that is a good point for strings to be applied for describing the physics of hadrons. However, the growth is so rapid that the partition function of a string gas converges only for temperatures below $T_{\rm H}$, which indicates the breakdown of the string picture at higher T. There were many attempts to treat $T_{\rm H}$ as a temperature associated with a phase transition. The idea is that at low T the QCD spectrum may be described by strings and above $T_{\rm H}$ by a gas of quarks and gluons.

However, the recent results from lattice QCD at zero baryonic chemical potential μ_B indicate that in the real world there is no phase transition but an analytic crossover [3].[1] This changes the story completely. If strings are indeed relevant for QCD then one has to show that a stringy description is also valid for high T.

In this paper, we propose two possible models for getting the high temperature behavior of strings which looks like that in QCD. Good motivations for this are models which recently helped to resolve another somewhat similar puzzle: How does string theory reproduce the hard behavior of gauge theory scattering amplitudes? In particular, in [5] an elegant solution based on Maldacena duality [6] was suggested. The essence of this solution is the warped geometry of the string dual. Later, in [7], string models with quantized tension were proposed to obtain string amplitudes with Regge poles and parton behavior. So we are bound to learn something if we succeed.

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[1]It is believed that this is the case for nonzero values of μ_B up to a few hundred MeV. For a review, see [4].
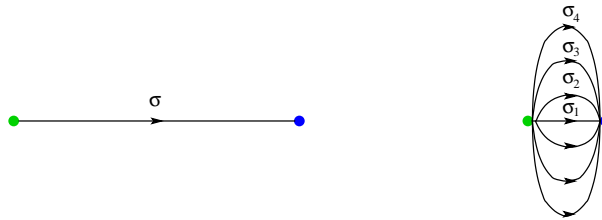
Figure 1. Left: a typical string-like flux tube of tension σ. Right: a fat string as a collection of thin strings with different tensions.

2 Possible resolutions

2.1 Flux tube picture

There is a long history, going back to Nambu's conjecture, of modeling QCD-like states using flux tubes. In particular, extensive numerical simulations have demonstrated that the flux picture does occur: a string-like chromoelectric flux tube forms between distant static color charges, as shown in figure 1 at left.[2] On the other hand, it turns out that QCD at short distances is in violent disagreement with expectations from a fluctuating string. The common wisdom is that, in order for the flux picture to be of use, the flux must be much longer than its intrinsic width. Since at short distances a string, as observed in lattice QCD, gets fat, this completely destroys the string picture.

Before we look at more detailed modeling in QCD, let us begin by thinking about the flux tubes in QED. The Faraday's concept of lines of flux emanating from charged bodies is a nice way to visualize electric and magnetic fields. For two opposite charges, it looks like that shown in figure 1 at right. If we assume that such a picture holds in QCD at short distances, then we come to a conclusion that fluxes of different length should have different tensions. Otherwise, we end up with a single flux shown in figure 1 at left which has the minimal energy. The next step is to interpret the fat string of lattice QCD as a set of thin strings of different tensions $\{\sigma_n\}$, as illustrated in figure 1 at right.

Having established a starting point for the flux picture at short distances, we turn our attention to more detailed properties of it. First, in the case of continuous spectrum of string tensions it seems natural to promote σ to a new spacetime coordinate. If so, then strings effectively live in some warped 5-dimensional space with σ playing a role of the fifth dimension. In fact, it was Polyakov who first realized that in order to describe 4-dimensional gauge theories dual string theories must be 5-dimensional [9]. Note that in doing so, he interpreted the Liouville field as the fifth coordinate and introduced the notion of running tension. In contrast, we used a more phenomenological way based on the Coulomb-type heavy quark potential at short distances.[3] Second, another possibility is a discrete spectrum of string tensions. This line of thinking was pursued in [7].[4] Certainly,

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[2]See, e.g., [8] for a review.
[3]We are grateful to V.I. Zakharov for numerous discussions of these issues.
[4]An earlier proposal of a dual resonance model, but with quantized Regge slope, is due to [10].

this is not the whole story. Spectra of more realistic models might contain both discrete and continues pieces.

2.2 Quantized tension

We begin with an ensemble of strings whose tensions take discrete values (quantized) [7]. This implies that the original string parameter α' is promoted to a discrete function α'_n with n a positive integer. The explicit form of α'_n is restricted by requiring that the mass spectrum of the n-th string is a subset of that of the primary string. The simplest possible proposal for the free energy of such an ensemble seems to be that F is a sum of string free energies[5]

$$F = \sum_{n=1}^{\infty} w_n F(n) \,. \tag{2.1}$$

Here the first term $F(1)$ is associated with a primary string such that $\alpha'_1 = \alpha'$ and $w_1 = 1$. Then, the contribution of the n-string $F(n)$ is obtained from $F(1)$ by replacing $\alpha' \to \alpha'_n$.

To simplify things slightly and illustrate the essential point, we will consider $F(1)$ as a partition function of the free string gas and choose $\alpha'_n = \alpha'/\mathcal{P}_k(n)$ with $\mathcal{P}_k(n)$ being a polynomial of degree k such that $\mathcal{P}_k(n)$ takes only integer values and grows with n. With this choice, we have for higher states[6]

$$\sum_n w_n \int^{\infty} [dh] \, \mathrm{e}^{4\pi h} \mathrm{e}^{-h\sqrt{\mathcal{P}_k}/\sqrt{\alpha'}T} \,. \tag{2.2}$$

The first term of the series diverges for temperatures greater than the Hagedorn temperature $T_{\mathrm{H}} = 1/4\pi\sqrt{\alpha'}$. The divergence occurs because the density of states (entropy factor) grows exponentially. This is the well known story. The novelty is that now one can get rid of it by imposing a new lower bound for the spectrum of the string tensions. Indeed, if

$$n_* < n \,, \tag{2.3}$$

with n_* a solution to the equation

$$\mathcal{P}_k(n) = T^2/T_{\mathrm{H}}^2 \,, \tag{2.4}$$

then one finds that the exponent in (2.2) is negative and the integral over h is convergent. Notice that there is no effect on the lower bound if $T < T_{\mathrm{H}}$ because $\mathcal{P}_k(1) = 1$. Also, (2.3) has a simple physical meaning: it suppresses long strings in the ensemble which are responsible for the divergence and, as a result, no limiting temperature exists.

This discussion suggests that we have to refine the proposal for the free energy. In the present case, we can take

$$F = \sum_{\max\{1, 1+[n_*]\}}^{\infty} w_n F(n) \,, \tag{2.5}$$

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[5]This is similar in spirit to what was proposed for scattering amplitudes in [7]. Note that it assumes no interaction between strings with different tensions.

[6]Here, for simplicity, we consider the bosonic case.

where $[n_*]$ means the integer part of n_*.

Now we will investigate the behavior of F at high temperature. In this case, eq.(2.4) simplifies to $a_k n^k = T^2/T_{\rm H}^2$ which has a solution $n_* = \sqrt[k]{T^2/a_k T_{\rm H}^2}$. We can replace the sum by an integral and $[n_*]$ by n_* that is also applicable for large n. Assuming that w_n is a product of power functions like $n^\delta \mathcal{P}_k^\gamma$, eq.(2.5) then becomes

$$F = \frac{1}{\sqrt{\alpha'}} \int_{n_*}^\infty dn\, n^\delta \mathcal{P}_k^\gamma\, F\big(\sqrt{\alpha'/\mathcal{P}_k}\, T\big)\,. \tag{2.6}$$

We have included the factor $1/\sqrt{\alpha'}$ on dimensional grounds. To see that the integral provides the desired power-law behavior, we rescale n as $n \to \sqrt[k]{T^2/T_{\rm H}^2}\, n$. As a result, we have[7]

$$F = F_0 T^{2\gamma+2(\delta+1)/k}\Big[1 + O\Big(\frac{1}{T}\Big)\Big]\,. \tag{2.7}$$

The parameters can be fixed by fitting the high temperature behavior in QCD. We take the option $\delta = -1$ and $\gamma = 2$. It is worth noting that these values also lead to the correct partonic behavior of scattering amplitudes [7][8] that provides a cross check of the model.

The last issue concerning the string model with the quantized tension that we will discuss here is the low temperature behavior. If the primary string has no tachyon and massless modes in its spectrum (or, in other words, it has a mass gap), then for low T the free energy of the n-th string is given by $F(n) \sim T^a \mathrm{e}^{-m_n/T}$ with $m_n \sim \mathcal{P}_k(n)/\sqrt{\alpha'}$. Since $\mathcal{P}_k$ is an increasing function of n, the sum (2.5) is dominated by the term with $n = 1$, and as a result, the free energy exponentially decreases at low temperature[9]

$$F = A\mathrm{e}^{-\frac{m}{T}}\,, \tag{2.8}$$

where $m = m_1$.

Thus, in the two different limits we have modeled precisely the temperature behavior as that of QCD at zero baryonic chemical potential.

2.3 Warped geometry

Since a string dual of QCD is unknown, for illustrative purposes, we consider a string theory whose space-time is a product of the Schwarzschild black hole in AdS_5 with a compact five-dimensional space X

$$ds^2 = \frac{r^2}{R^2}\big(l dt^2 + d\vec{x}^2\big) + \frac{R^2}{r^2}l^{-1}dr^2 + ds_{\rm X}^2\,, \quad l = 1 - \frac{r_{\rm T}^4}{r^4}\,, \quad r_{\rm T} = \pi R^2 T\,, \tag{2.9}$$

where t is periodic with period $\beta = 1/T$. It is believed that such a black hole geometry does capture important properties of QCD at high T [11]. Because of the warping, in local inertial coordinates the period, for a state approximately localized in the r-direction, is given by

$$\tilde{\beta} = \sqrt{l}\frac{r}{R}\,\beta\,, \quad \text{with} \quad r_{\rm T} < r < \infty\,. \tag{2.10}$$

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[7]Note that F_0 must be finite, otherwise (2.7) makes no sense.

[8]Note that the shift $\gamma \to \gamma + 2$ is due to scattering of vector particles in [7].

[9]We will not need the detail expression for A in this discussion.

The essential point is that a single ten-dimensional temperature $\tilde{T} = 1/\tilde{\beta}$ can give rise to different values of a four-dimensional temperature T. Indeed, for given $\tilde{\beta}$, (2.10) becomes

$$T = \alpha r \,, \tag{2.11}$$

with $\alpha^2 = -\frac{1}{2}\tilde{\beta}^2 + \sqrt{\pi^4 R^4 + \frac{1}{4}\tilde{\beta}^4}/\pi^4 R^6$. We see that T grows with r and approaches infinity for $r \to \infty$. This allows us to study the string free energy at high T even for low ten-dimensional temperature such that $\tilde{T} < T_{\mathrm{H}}$. The latter, for example, means that the partition function of the free string gas remains finite and no limiting temperature occurs. Thus, the warping might be a point in describing the crossover of QCD in terms of strings.[10]

To write an expression for the string free energy on backgrounds like (2.9) in the large-r region, we treat it as a ten-dimensional expression at fixed r, integrated coherently over this position[11]

$$\frac{F}{T} = -\ln Z = \frac{1}{\alpha'^5} \sum_{i=1}^{\infty} g_{\mathrm{eff}}^{2(i-1)} \int d^{10}x \sqrt{g}\, F^{(i)}(\sqrt{\alpha'}\tilde{T}) \,, \tag{2.12}$$

where g_{eff} is an effective coupling and $F^{(i)}$ is a i-loop contribution [2]. We have included the factor $\frac{1}{\alpha'^5}$ on dimensional grounds. Using (2.9) and (2.10), we find the free energy density

$$f = \frac{F}{V} = \frac{1}{\alpha'^5} \frac{V_{\mathrm{X}}}{R^3} \sum_{i=1}^{\infty} g_{\mathrm{eff}}^{2(i-1)} \int_{\pi R^2 T}^{\infty} dr\, r^3\, F^{(i)}\big(\sqrt{\alpha'}RT/\sqrt{l}r\big) \,, \tag{2.13}$$

where V is a $3d$ spatial volume and V_{X} is a volume of the internal space. The integration limits are due to (2.10). From this expression it is evident that a simple rescaling $r \to Tr$ leads to

$$f \sim T^4 \,. \tag{2.14}$$

The scaling of this free energy density with T is precisely as in QCD.

Now let us look at the lower limits of the integrals in (2.6) and (2.13). In (2.6), n_* comes from the bound (2.3) which suppresses long strings and yields the convergence of the partition function of the free string gas; the lower limit in (2.13) is due to the horizon of the black hole (2.9). Does it mean that the horizon suppresses long strings? We can gain some understanding of this by choosing $\mathcal{P}_2(n) = n^2$ and discretizing the r-direction as $r = nR$. Then, (2.3) can be rewritten as $T/T_{\mathrm{H}} < n$. On the other hand, $r_{\mathrm{T}} < r$ now becomes $c(\lambda)T/4T_{\mathrm{H}} < n$, where $c = R/\sqrt{\alpha'}$ is a function of the 't Hooft coupling λ. We see that the horizon resolves the Hagedorn singularity if the coupling is large enough, such that $c(\lambda) > 4$.[12]

We conclude this section by making a few remarks. First, our derivation assumes that the right hand side of (2.13) is finite. Even if we avoid the problem of the Hagedorn

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[10]Aside from hard scattering [5], where a single ten-dimensional momentum $\tilde{p}$ also gives rise to different values of a four-dimensional momentum p via $p = \frac{\tilde{p}}{R}r$, the warping is also the essence of the Randall-Sundrum proposal for solving the hierarchy problem [12].

[11]Such a treatment seems natural for the states localized in the r-direction. In the context of scattering amplitudes, it was proposed in [5].

[12]Formally, we may consider only the term $F^{(1)}$ in (2.13).

singularity in ten dimensions, there is no guarantee for convergence. Unfortunately, we cannot, with our present methods, perform accurate calculations in type IIB string theory on AdS$_5$ and, therefore, address the issue of convergence. Second, it is straightforward to extend the above analysis to any background whose metric is given by (2.9) at large r and, as a result, recover the desired scaling of the free energy at high T. Unfortunately, there is no satisfactory modeling for a string background which is dual to QCD at low T. So, we leave the analysis of this limit for future study. Finally, as in the case of high-energy string scattering in warped spacetime [13], the power law behavior at high temperature can be understood from the analysis of bosonic zero modes in a string path integral. These modes contribute the volume factor $\int d^{10}\sqrt{g}$ which does lead to the scaling of the free energy with temperature like in QCD.

3 Concluding comments

Although at first glance, both approaches look similar, they are really different. In the first approach the starting point is an ensemble of strings which has already a good behavior at low T. There is no limiting (Hagedorn) temperature because long strings are suppressed at high T. In the second approach one gets high T in a gauge theory from a ten-dimensional string dual whose temperature $\tilde{T}$ is below $T_{\rm H}$. The effect is due to the warping, where a single ten-dimensional temperature gives rise to many different values of a four-dimensional temperature and, as a result, there is no need to go beyond $T_{\rm H}$ on the string theory side to get high T on the gauge theory side.

It is necessary to stress that our analysis goes beyond the supergravity approximation. It is quite general and may be also applied to other supersymmetric and heterotic string theories.

For large N_c, a gauge theory free energy is of order N_c^2. If the sum in (2.12) is an $1/N_c$ expansion, then it has no contribution from zero genus as it starts from genus one. This seems puzzling. A possible resolution is that zero genus somehow effectively occurs after resummation. To see that this is indeed the case, we need the full control of type IIB string theory on curved backgrounds like AdS$_5$ that is unfortunately beyond our grasp at present.

As in [5, 7], we have been able to recover the partonic behavior at high energy (temperature) without revealing the nature of partons as well as the logarithmic scaling violation. A more complete description remains an interesting avenue for future research.

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